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ON THE USE OF THE COMPLEMENTARY ENERGY IN THE SOLUTION OF BUCKLING PROBLEMS†

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Abstract—A systematic derivation of the expression for the complementary energy in elastic buckling problems is presented. Compatibility is identified with variation with respect to the stress components, and the resulting eigenvalue problem is shown to be equivalent to, and sometimes more convenient than, the corresponding formulation in terms of the potential energy. Similarly, approximate techniques may lead to better as well as simpler estimates, whose upper bound property can, however, be assured only through appropriate safeguards.

The method is applied in some detail to buckling of columns of arbitrary boundary conditions and axial force distribution. Another example is the problem of lateral beam buckling, with the effect of warping restraint included. In both cases (and presumably in many others) the complementary energy formulation is of lower order than the conventional potential energy formulation, and it is clear that the same simplification should also apply to finite elements or other discrete formats. The method is restricted to the (technically significant) case of a linear prebuckling state.

1. INTRODUCTION

The connection between structural stability and potential energy is well established and fundamental. Under conservative loading conditions the equilibrium of an elastic structure is stable, that is, the response of the structure to small disturbances is also small, if the potential energy associated with the state of equilibrium is less than that of any neighboring configuration. Conversely, the structure may buckle if this is not the case. Since the expression for the potential energy in the neighborhood of an equilibrium configuration is dominated by the term (hereafter called "the potential energy" Ω) which is quadratic in the displacements, the "critical load" is identified by the condition that the smallest possible value of Ω must vanish. It is also well known that an analysis of the post-critical behavior, and hence of the imperfection sensitivity, of the structure requires the inclusion of higher order terms in the expression for the potential energy. This concept, which was first proposed by Karman and Tsien[1], has been explored systematically by Koiter[2] and dates back to an initial conjecture by Donnell[3].

There are a number of methods of computing the critical load. Again in the case of a conservative loading condition, these methods, including the method of potential energy, are all equivalent and lead to the same result. The practical and numerical significance of the potential energy method lies in the relative ease with which it is adaptable to obtaining an approximate solution to a complicated and otherwise intractable problem. Executed properly, whether in the form of a Rayleigh fraction, a Galerkin approximation, a finite element discretization, etc., it leads to upper bounds to the critical load, and these bounds can be made to approach the exact value in a straightforward manner.

Compared with the potential energy, the role of the complementary energy is neither as well established nor apparently as fundamental. The problem of structural instability, which involves the superposition of disturbances on an existing equilibrium configuration, is inherently nonlinear in the strain displacement relations. The inclusion of this nonlinearity in the potential energy formulation presents no conceptual difficulties and leads, through the variational process, directly to the equations of equilibrium in the deformed (rather than in the original) configuration.

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Similarly the complementary energy leads to the compatibility conditions if only states of stress which are in equilibrium are admitted in competition. Unfortunately, relative to the deformed configuration, these equations of equilibrium contain not only the stresses, but also the displacements if a material (i.e. Lagrangian) approach is used, as is done in most instability studies, including the present one. A deceptively simple way of avoiding this difficulty is the introduction of the spatial (Eulerian) stress components. However, in this case new difficulties arise in connection with the constitutive relations. No generally useful variational principle involving the stresses alone appears to have been established thus far.

Then why consider a complementary energy formulation at all? A well-known supportive example is the case of a simply supported column under axial compression. An alternative to the Rayleigh fraction (via the potential energy) is the Timoshenko fraction (via the complementary energy), and the latter, which is also an upper bound to the critical load, is known to be smaller than the former and hence a better approximation.

An additional, though probably not independent, advantage of pursuing a complementary energy formulation is the fact that it often involves equations (exact or approximate) whose order is less than the one associated with a potential energy formulation. Since the stresses, rather than the displacements, are now the variables, it becomes possible to deal with fewer boundary conditions and with simpler continuity conditions at points of design discontinuity, and although this question is not explored in the current study it appears likely that a finite element approximation permits the establishment of simpler elements and simpler shape functions. The total computational effort may therefore be reduced, notwithstanding an improvement in the accuracy.

These advantages appear to have been first pointed out by Grammel[4]†, who posed the closely related beam vibration problem in terms of a Galerkin procedure in conjunction with approximate deflection functions which are obtained in the course of the first cycle of a standard iteration (equivalent to satisfying the equations of equilibrium). Schaefer[5] treated the problem of nonlinear beam bending, including buckling, by applying a canonical transformation to the conventional variational formulation based on the potential energy and by once again postulating the satisfaction of the equations of equilibrium. It can readily be shown that both investigations, and others of a similar nature, are special cases of complementary energy formulations, although in somewhat disguised form.

More explicitly tied to complementary energy is the work of Oran[6–8], who extended the concept of the Timoshenko fraction to problems other than the original one of a simply supported column. Oran has found, however, that the approximate results so obtained are not necessarily upper bounds nor good approximations. Popelar[9] has traced this erratic behavior to the existence of singular solutions, associated with vanishing loads, in the case of statically indeterminate structures, and to the need to consider only functions which are orthogonal to these singular solutions.

In what follows a systematic formulation of the complementary energy principle and its application to buckling problems is presented. Section 2 contains a general derivation, and Sections 3 and 4 the application to the problem of column buckling and of torsional buckling, respectively, for arbitrary boundary conditions. A straightforward extension to vibration problems (and, by implication, to other eigenvalue problems) is contained in Section 5, together with some concluding remarks and observations. It is noted that both theory and examples are based on the assumption of a linear prebuckling state.

2. GENERAL DERIVATION

Since the discussion deals with structural problems, rather than with problems involving a three-dimensional continuum, it is convenient to use the notation introduced by Budiansky and Hutchinson[10]. Accordingly we identify the components of a generalized displacement vector by $\mathbf{u}$ and those of a generalized stress and strain vector by $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, respectively. Let λ be a common multiplier of all applied surface tractions and body forces (referred to as "loads"), and let $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0$ represent an equilibrium state. Then the equations of equilibrium may be expressed in the compact form

$$\boldsymbol{\sigma}_0{}^T \delta\boldsymbol{\varepsilon} = \lambda W(\delta\mathbf{u}) \tag{1}$$

†The authors are indebted to one of the referees for bringing [4] and [5] to their attention.

in which $\boldsymbol{\sigma}_0^T$ is the transpose of $\boldsymbol{\sigma}_0$, W is the work, for $\lambda = 1$, of the loads relative to an arbitrary displacement variation $\delta\mathbf{u}$, and the left side of eqn (1) is to be interpreted as representing the integral over the total domain of the independent variables (a single variable for beams and columns, two variables for plates, shells, etc.).

Equation (1) identifies equilibrium by equating the internal work with the external work during a virtual displacement. We postulate here that the external work is linear in the displacements. Although this is usually true there are some counterexamples, such as that of a ring subjected to given hydrostatic pressure. In that case, a suitable adjustment can be made without difficulty.

The kinematics of the problem is established by letting

$$\boldsymbol{\varepsilon} = \mathbf{l}_1(\mathbf{u}) + \frac{1}{2}\mathbf{l}_2(\mathbf{u}) \tag{2}$$

in which $\mathbf{l}_1$ and $\mathbf{l}_2$ are linear and quadratic, respectively, in the displacements. Hence, for $\mathbf{u} = \mathbf{u}_0$, insertion of eqn (2) into eqn (1) leads to

$$\boldsymbol{\sigma}_0^T[\mathbf{l}_1(\delta\mathbf{u}) + \mathbf{l}_{11}(\mathbf{u}_0\delta\mathbf{u})] - \lambda W(\delta\mathbf{u}) = 0 \tag{3}$$

for all virtual displacements which exhibit the required degree of continuity and which satisfy the kinematic boundary conditions

$$\mathbf{B}(\delta\mathbf{u}) = 0 \tag{4}$$

on that part of the boundary on which the displacements are prescribed.

The critical buckling load is reached when bifurcation in the equilibrium path occurs. That is, let a neighboring state defined by

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + \dot{\boldsymbol{\sigma}}, \quad \mathbf{u} = \mathbf{u}_0 + \dot{\mathbf{u}}, \quad \mathbf{B}(\dot{\mathbf{u}}) = 0 \tag{5}$$

satisfy eqns (3) and (4), subject to the same load parameter λ. Then, after linearization with respect to the additional quantities,

$$\dot{\boldsymbol{\sigma}}^T[\mathbf{l}_1(\delta\mathbf{u}) + \mathbf{l}_{11}(\mathbf{u}_0\delta\mathbf{u})] + \boldsymbol{\sigma}_0^T\mathbf{l}_{11}(\dot{\mathbf{u}}\delta\mathbf{u}) = 0 \tag{6}$$

represents the critical condition.

In the case of an elastic body we postulate the existence of a strain-energy density $U(\boldsymbol{\varepsilon})$ such that

$$\boldsymbol{\sigma}_0 = \left(\frac{\mathrm{d}U}{\mathrm{d}\boldsymbol{\varepsilon}}\right)_0 \tag{7}$$

and hence

$$\dot{\boldsymbol{\sigma}} = \left[\frac{\mathrm{d}^2U}{\mathrm{d}\boldsymbol{\varepsilon}^2}\right]_0 \dot{\boldsymbol{\varepsilon}} \equiv [A]\dot{\boldsymbol{\varepsilon}}. \tag{8}$$

Then, in view of eqn (8), eqn (6) is equivalent to the variational problem

$$\begin{aligned} &\Omega \equiv \frac{1}{2}\{\boldsymbol{\sigma}_0^T\mathbf{l}_2(\dot{\mathbf{u}}) + \dot{\boldsymbol{\varepsilon}}^T[A]\dot{\boldsymbol{\varepsilon}}\} \\ &\dot{\boldsymbol{\varepsilon}} = \mathbf{l}_1(\dot{\mathbf{u}}) + \mathbf{l}_{11}(\mathbf{u}_0\dot{\mathbf{u}}) \qquad \mathbf{B}(\dot{\mathbf{u}}) = 0 \\ &\delta_{\dot{\mathbf{u}}}\Omega = 0. \end{aligned} \tag{9}$$

As noted in the introduction, Ω represents the dominant (i.e. quadratic) term in the potential energy expansion.

Most structures exhibit insignificant displacements, though not stresses, prior to buckling†,

†A typical counterexample is that of a shallow arch or shell cap. In this case the approximation of eqn (10) leads to significant errors.

and it is therefore permissible and customary to let

$$\mathbf{u}_0 = \mathbf{0} \qquad \boldsymbol{\sigma}_0 = -\lambda\boldsymbol{\tau}. \tag{10}$$

In that case eqns (9) can be summarized compactly by

$$\Omega^{***} \equiv \frac{1}{2}\{\lambda\boldsymbol{\tau}^T l_2(\dot{\mathbf{u}}) - \dot{\boldsymbol{\varepsilon}}^T[A]\dot{\boldsymbol{\varepsilon}}\} + \dot{\boldsymbol{\sigma}}^T\{\dot{\boldsymbol{\varepsilon}} - \mathbf{l}_1(\dot{\mathbf{u}})\} - \dot{\mathbf{r}}^T\mathbf{B}(\dot{\mathbf{u}})$$
$$\underset{\dot{\mathbf{u}},\boldsymbol{\varepsilon},\dot{\boldsymbol{\sigma}},\dot{\mathbf{r}}}{\delta}\ \Omega^{***} = 0 \tag{11}$$

in which $\dot{\boldsymbol{\sigma}}$ and $\dot{\mathbf{r}}$ have been introduced as Lagrangian multipliers. Variation of eqn (11) with respect to $\dot{\boldsymbol{\varepsilon}}$ leads to eqn (8), and when this is inserted in eqn (11) we obtain

$$\Omega^{**}_{(\dot{\mathbf{u}}\dot{\boldsymbol{\sigma}}\dot{\mathbf{r}})} = \frac{1}{2}\{\dot{\boldsymbol{\sigma}}^T[C]\dot{\boldsymbol{\sigma}} + \lambda\boldsymbol{\tau}^T\mathbf{l}_2(\dot{\mathbf{u}})\} - \dot{\boldsymbol{\sigma}}^T\mathbf{l}_1(\dot{\mathbf{u}}) - \dot{\mathbf{r}}^T\mathbf{B}(\dot{\mathbf{u}}) \tag{12}$$

in which $[C] = [A]^{-1}$ is the compliance density.

With the substitution

$$\lambda\dot{\mathbf{u}} = \mathbf{v} \tag{13}$$

(and after deleting the dot superscripts on the additional quantities) we now express $\mathbf{v}$ in terms of $\boldsymbol{\sigma}$ and $\mathbf{r}$ by virtue of the equation of equilibrium

$$\lambda\delta_{\mathbf{v}}\Omega^{**} = \boldsymbol{\tau}^T\mathbf{l}_{11}(\mathbf{v}\delta\mathbf{v}) - \boldsymbol{\sigma}^T\mathbf{l}_1(\delta\mathbf{v}) - \mathbf{r}^T\mathbf{B}(\delta\mathbf{v}) = \mathbf{0}. \tag{14}$$

Substitution of $\mathbf{v}$ into eqn (12) then completes the determination of the complementary energy Ω^* as a function of the generalized stresses $\boldsymbol{\sigma}$ and the reactions $\mathbf{r}$.

The process is simplified somewhat by employing eqn (14) (with $\delta\mathbf{v} = \mathbf{v}$) and eqn (13). Substitution of these equations in eqn (12) results in the simple form

$$\lambda\,\Omega^*_{(\sigma,r)} = \frac{1}{2}\{\lambda\boldsymbol{\sigma}^T[C]\boldsymbol{\sigma} - \boldsymbol{\tau}^T\mathbf{l}_2(\mathbf{v})\} \tag{15}$$

in which once again $\mathbf{v}$ is a function of $\boldsymbol{\sigma}$ and $\mathbf{r}$ through eqn (14). The variational equations

$$\delta_\sigma\Omega^* = [C]\boldsymbol{\sigma} - \frac{1}{\lambda}\mathbf{l}_1\{\mathbf{v}_{(\sigma,r)}\} = \mathbf{0}$$
$$\delta_{\mathbf{r}}\Omega^* = -\frac{1}{\lambda}\mathbf{B}\{\mathbf{v}_{(\sigma,r)}\} = \mathbf{0} \tag{16}$$

then yield the conditions of compatibility.

If the problem is n times kinematically overconstrained (or, equivalently, n times statically indeterminate), the equilibrium equations arising out of eqn (14) have nontrivial eigensolutions $\boldsymbol{\sigma}_i$, $\mathbf{r}_i$ $(i = 1, 2, \ldots, n)$ corresponding to $\lambda = 0$ and satisfying

$$\boldsymbol{\sigma}_i^T\mathbf{l}_1(\delta\mathbf{v}) + \mathbf{r}_i^T\mathbf{B}(\delta\mathbf{v}) = \mathbf{0} \qquad (i = 1, 2, \ldots, n). \tag{17}$$

These eigensolutions, however, violate compatibility since, for $\lambda = 0$, the second term on the right side of eqn (15) is not present, and eqn (16), after corresponding mutilation, then no longer constitute the compatibility conditions.

As has been pointed out by Popelar[9] the smallest nonvanishing buckling load or, correspondingly, an upper bound to it through an approximate technique, is obtained if only solutions are admitted in competition which are orthogonal to these eigensolutions. The form of this orthogonality is found by substituting eqn (16) in eqn (17) (with $\delta\mathbf{v} = \mathbf{v}$). This process leads to the system of conditions

$$\boldsymbol{\sigma}^T[C]\boldsymbol{\sigma}_i = 0 \qquad (i = 1, 2, \ldots, n). \tag{18}$$

By eqn (14) $\mathbf{v}$ is linearly related to $\boldsymbol{\sigma}$ and $\mathbf{r}$. Since $[C]$ is positive definite, eqns (15) and (16) may be expressed in the equivalent form

$$\lambda_{cr} = \min_{\boldsymbol{\sigma},\mathbf{r}} \lambda_T \equiv \min_{\boldsymbol{\sigma},\mathbf{r}} \frac{\boldsymbol{\tau}^T \mathbf{l}_2(\mathbf{v})}{\boldsymbol{\sigma}^T[C]\boldsymbol{\sigma}} \tag{19}$$

provided that the numerator is also positive definite. In eqn (19) $\mathbf{v}$ is once again expressed in terms of $\boldsymbol{\sigma}$ and $\mathbf{r}$ by means of eqn (14), and the stress $\boldsymbol{\sigma}$ is restricted through eqns (18). In order to incorporate these restrictions it is convenient, and in view of the positive definiteness of $[C]$ always possible, to orthonormalize the system of eigenstresses, $\boldsymbol{\sigma}_i$, by letting

$$\boldsymbol{\sigma}_i^T[C]\boldsymbol{\sigma}_j = \delta_{ij} \qquad (i, j = 1, 2, \ldots, n). \tag{20}$$

Then the restriction embodied in eqn (18) is removed by identifying λ_T through

$$\lambda_T = \frac{\boldsymbol{\tau}^T \mathbf{l}_2(\mathbf{v})}{\boldsymbol{\sigma}^T[C]\boldsymbol{\sigma} - \sum_{i=1}^{n} \{\boldsymbol{\sigma}^T[C]\boldsymbol{\sigma}_i\}^2} \tag{21}$$

in which the denominator continues to be non-negative by Bessel's inequality.

We finish our general discussion by noting that the conventional Rayleigh fraction, which is based on the potential energy principle, is given by

$$\lambda_R = \frac{\mathbf{l}_1^T(\mathbf{v})[A]\mathbf{l}_1(\mathbf{v})}{\boldsymbol{\tau}^T \mathbf{l}_2(\mathbf{v})}. \tag{22}$$

It has been shown by Popelar[11] that for the same assumed buckling mode $\mathbf{v}$ (and associated stress systems $\boldsymbol{\sigma}$ via eqn (14))

$$\lambda_{R_{\min}} \geqq \lambda_{T_{\min}} \geqq \lambda_{cr}. \tag{23}$$

Inequality (23) shows that λ_T, which represents a generalization of the familiar Timoshenko fraction associated with the buckling of simply supported columns, is a better approximation than the Rayleigh fraction λ_R. Popelar's proof is based on the definition of λ_T in terms of eqn (19), but an extension to the more general form of eqn (21) is straightforward.

3. COLUMN BUCKLING

In this section we apply the general formulation developed in the previous section to the problem of the instability of a column which is subjected to axial forces. Figure 1 shows a column which is simply supported at one end and fixed at the other, but it will appear obvious from the text that the method is equally applicable to any other set of boundary conditions. A set of distributed axial forces (representing, for example, the weight of the column itself) is also included for the sake of completeness.

In this case the components of the generalized stress vector are the axial force N and bending moment M, and those of the strain vector the axial strain ε and curvature κ. That is,

$$\begin{aligned} &\boldsymbol{\sigma} = \begin{Bmatrix} N \\ M \end{Bmatrix} \qquad \boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon \\ \kappa \end{Bmatrix} = \frac{1}{\lambda}\begin{Bmatrix} w' \\ v'' \end{Bmatrix} + \frac{1}{2\lambda^2}\begin{Bmatrix} v'^2 \\ 0 \end{Bmatrix} \\ &\boldsymbol{\tau} = \begin{Bmatrix} N_0 \\ 0 \end{Bmatrix} \qquad [C] = \begin{bmatrix} \dfrac{1}{EA} & 0 \\ 0 & \dfrac{1}{EI} \end{bmatrix} \left(' \equiv \frac{\mathrm{d}}{\mathrm{d}z} \right) \end{aligned} \tag{24}$$

in which w and v are the displacements in the axial and lateral direction, respectively, and the

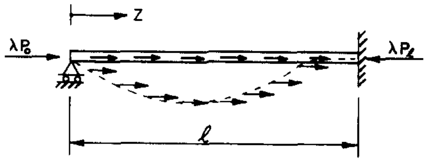

Fig. 1. Typical column.

diagonal form of $[C]$ is achieved by embedding the coordinate system in the centroid of the cross section.

With the boundary conditions

$$v_{(0)} = v_{(l)} = v'_{(l)} = w_{(l)} = 0 \tag{25}$$

eqn (12), subject to eqn (13), takes the form

$$\lambda\Omega^{**} = \frac{1}{2}\int_0^l \left[\lambda\left(\frac{N^2}{EA}+\frac{M^2}{EI}\right) + N_0 v'^2 - 2Nw' - 2Mv''\right] dz$$
$$+ q_0 v_{(0)} - q_l v_{(l)} + m_l v'_{(l)} + n_l w_{(l)}, \tag{26}$$

and hence eqn (14) corresponds to

$$\lambda\delta_w\Omega^{**} = \int_0^l N'\delta w \, dz + [n_l - N_{(l)}]\delta w_{(l)} - N_{(0)}\delta w_{(0)} = 0 \tag{27}$$

$$\lambda\delta_v\Omega^{**} = -\int_0^l [(N_0 v')' + M'']\delta v \, dz + [M_{(l)} - m_l]\delta v'_{(l)}$$
$$+ M_{(0)}\delta v'_{(0)} + [M'_{(l)} - q_l]\delta v_{(l)} - [M'_{(0)} - q_0]\delta v_{(0)} = 0. \tag{28}$$

It follows from eqn (27) that

$$N_{(z)} \equiv 0 \qquad n_l = 0. \tag{29}$$

Equation (28) implies

$$M'' + (N_0 v')' = 0 \qquad (0 \leqq z \leqq l) \tag{30}$$
$$m_l = M_{(l)} \qquad q_l = M'_{(l)} \qquad q_0 = M'_{(0)} \tag{31}$$
$$M_{(0)} = 0, \tag{32}$$

and hence, by integration of eqn (30),

$$M' + N_0 v' = Q = \text{const.} \tag{33}$$

Then, according to eqn (15),

$$\Omega^* = \frac{1}{2}\int_0^l \left[\frac{M^2}{EI} - \frac{1}{\lambda N_0}(M' - Q)^2\right] dz, \tag{34}$$

and therefore, in view of eqn (32),

$$\delta_M\Omega^* = 0: \qquad \lambda\frac{M}{EI} + \left(\frac{M' - Q}{N_0}\right)' = 0 \qquad (0 \leqq z \leqq l)$$
$$M'_{(l)} - Q = 0, \tag{35}$$

$$\delta_Q\Omega^* = 0: \qquad \int_0^l \frac{1}{N_0}(M' - Q)\, dz = 0. \tag{36}$$

In the context of column buckling the problem shown in Fig. 1 has one degree of statical indeterminacy and exhibits therefore one eigensolution associated with $\lambda = 0$. This is obtained from eqns (30) (with $N_0 = 0$), (31), and (32), or else from eqns (35) (with $\lambda = 0$) and (36), and is given, before normalization, by

$$M_1 = Q_1 z \qquad m_l = Q_1 l \qquad q_0 = q_l = Q_1. \tag{37}$$

The orthogonality condition eqn (18) is therefore

$$\int_0^l \frac{Mz}{EI}\,\mathrm{d}z = 0 \tag{38}$$

and the Timoshenko fraction for this case takes the form

$$\lambda_T = \frac{\displaystyle\int_0^l \frac{1}{N_0}(M' - Q)^2\,\mathrm{d}z}{\displaystyle\int_0^l \frac{M^2}{EI}\,\mathrm{d}z - \frac{\left(\displaystyle\int_0^l \frac{Mz}{EI}\,\mathrm{d}z\right)^2}{\displaystyle\int_0^l \frac{z^2}{EI}\,\mathrm{d}z}} \tag{39}$$

subject to $M_{(0)} = 0$.

If, in particular, the distributed axial force is absent, that is, $P_0 = P_l = 1$ and $N_0(z) = 1$, then eqn (35), with eqns (32) and (36), and after elimination of Q, becomes

$$\begin{aligned} &M'' + \lambda\frac{M}{EI} = 0 \qquad (0 \leqq z \leqq l) && \text{(a)}\\ &M_{(0)} = 0 && \text{(b)}\\ &M_{(l)} - lM'_{(l)} = 0. && \text{(c)} \end{aligned} \tag{40}$$

This is a singular Sturm–Liouville problem, and eqn (38) is necessary to remove the singularity. With the inclusion of eqn (38), the last of eqns (40) becomes redundant.

Considering again the same special case, and after eliminating Q by means of eqn (36), the Timoshenko fraction is now given by

$$\lambda_T = \frac{\displaystyle\int_0^l M'^2\,\mathrm{d}z - \frac{M_{(l)}^2}{l}}{\displaystyle\int_0^l \frac{M^2}{EI}\,\mathrm{d}z - \frac{\left(\displaystyle\int_0^l \frac{Mz}{EI}\,\mathrm{d}z\right)^2}{\displaystyle\int_0^l \frac{z^2}{EI}\,\mathrm{d}z}} \tag{41}$$

subject again to $M_{(0)} = 0$. It follows from the Schwarz inequality that the numerator of eqn (41) is positive definite.

For constant EI the solution of the problem is of course well known. It is re-established in the present context by setting

$$M = A \sin\frac{\alpha z}{l} \qquad \left(\alpha^2 = \frac{\lambda l^2}{EI}\right),$$

which satisfies eqns (40a) and (40b) and in which, by either eqn (38) or (40c),

$$\sin\alpha - \alpha\cos\alpha = 0,$$

whose smallest nontrivial root is $\alpha_1 = 4{\cdot}49$.

An approximate solution may be obtained by letting

$$M = A\left(\frac{z}{l} - \frac{z^2}{l^2}\right)$$

which satisfies eqn (40b), as required, but not eqn (40c). Substitution in eqn (41) leads to

$$\lambda_T = \frac{80}{3}\frac{EI}{l^2} > \alpha_1{}^2\frac{EI}{l^2} = \lambda_{cr},$$

and the upper bound property of the method is demonstrated. It is interesting to note that if the second term in the denominator of eqn (41) is deleted, then

$$\lambda_T = 10\frac{EI}{l^2} < \lambda_{cr},$$

and the upper bound property is violated, as has been observed previously by Oran [8]. In general, letting

$$M = A\frac{z^2}{l^2} + B\frac{z^3}{l^3}$$

and minimizing λ_T with respect to A and B we obtain

$$\lambda_T = 21{\cdot}3\frac{EI}{l^2}$$

which represents a very close upper bound. Conversely, with the second term in the denominator in eqn (41) deleted, the same process leads to an eigenvalue which is almost zero.

4. TORSIONAL BUCKLING

In this section we apply the complementary energy method to the problem of determining the critical load under which a beam becomes unstable with respect to lateral buckling in bending and torsion. Figure 2 shows such a beam, with its left end simply supported and its right end fixed against all rotations as well as against warping. These boundary conditions are typical, and the application of the method to other boundary conditions is straightforward.

In this case the generalized stress and strain vectors and the compliance density are given as follows:

$$\begin{aligned}
\boldsymbol{\sigma}^T &= \{M_x, M_y \equiv M, M_z \equiv T, B\}\\
\boldsymbol{\varepsilon}^T &= \frac{1}{\lambda}\{v'', u'', \beta', \beta''\} + \frac{1}{2\lambda^2}\{2u''\beta, -2v''\beta, 0, 0\} \qquad (42)\\
\boldsymbol{\tau}^T &= \{-m(z), 0, 0, 0\}\\
[C]_{\text{diag}} &= \left[\frac{1}{EI_x}, \frac{1}{EI}, \frac{1}{GK}, \frac{1}{E\Gamma}\right].
\end{aligned}$$

In eqn (42) the components of the stress vector are the bending moments about the major and minor axes, the torsional moment, and the bimoment, and the components of the generalized strain vector are chosen accordingly, with v, u and β representing the displacements in the two principal directions and the twisting rotation, respectively. The diagonal elements of $[C]^{-1}$ are the two bending stiffnesses, the torsional stiffness, and the warping stiffness, respectively, and it is assumed, without substantial loss of generality, that the coordinate axes are embedded in the principal centroidal axes, that the center of shear coincides with the centroid, and that the loads are applied at the centroid. Extensions to other conditions, which may also involve nondiagonal elements in $[C]$, are accomplished without undue difficulty.

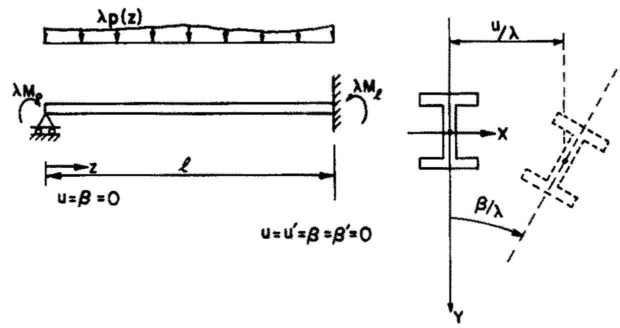

Fig. 2. Typical beam.

If Ω^{**} is established, according to eqn (12), and after variation with respect to v (leading to the elimination of v and of M_x), the remaining expression is as follows:

$$\lambda\Omega^{**}=\frac{1}{2}\int_0^l\left[\lambda\left(\frac{M^2}{EI}+\frac{T^2}{GK}+\frac{B^2}{E\Gamma}\right)-2Mu''-2T\beta'-2B\beta''-2mu''\beta\right]\mathrm{d}z$$
$$+q_0u_{(0)}-q_lu_{(l)}+m_lu'_{(l)}-t_0\beta_{(0)}+t_l\beta_{(l)}+b_l\beta'_{(l)}. \tag{43}$$

Variation of Ω^{**} with respect to u leads to

$$\begin{aligned}
&M''+(m\beta)''=0 \qquad (0\leqq z\leqq l) && \text{(a)}\\
&M_{(l)}+(m\beta)_{(l)}-m_l=0 && \text{(b)}\\
&-M_{(0)}-(m\beta)_{(0)}=0 && \text{(c)}\\
&-M'_{(l)}-(m\beta)'_{(l)}+q_l=0 && \text{(d)}\\
&M'_{(0)}+(m\beta)'_{(0)}-q_0=0 && \text{(e)}
\end{aligned} \tag{44}$$

which, after intergration, becomes

$$M+m\beta=Qz \qquad (0\leqq z\leqq l)$$
$$q_0=q_l=\frac{m_l}{l}=Q, \tag{45}$$

with Q representing once again an unknown static constant to be determined later. Similarly the β variation of Ω^{**} results in the system of equations

$$\begin{aligned}
&T'-B''-mu''=0 \qquad (0\leqq z\leqq l) && \text{(a)}\\
&-B_{(l)}+b_l=0 && \text{(b)}\\
&B_{(0)}=0 && \text{(c)}\\
&B'_{(l)}-T_{(l)}+t_l=0 && \text{(d)}\\
&-B'_{(0)}+T_{(0)}-t_0=0 && \text{(e)}
\end{aligned} \tag{46}$$

Then, if eqns (45) and (46) are incorporated in the general formulation of Ω^* in eqn (15), the complementary energy expression becomes

$$\Omega^*_{(M,T,B,Q)}=\frac{1}{2}\int_0^l\left[\left(\frac{M^2}{EI}+\frac{T^2}{GK}+\frac{B^2}{E\Gamma}\right)-\frac{2}{\lambda m}(M-Qz)(T'-B'')\right]\mathrm{d}z \tag{47}$$

subject, as in eqn (46c), to $B_{(0)}=0$.

The singular eigenfunctions are obtained, according to eqn (17), by setting $\lambda=0$ or, equivalently, by solving eqns (44) and (46) with $m=0$. With the boundary conditions employed in the present example this leads to two systems of eigenstresses defined by

$$M_1 = Q_1 z \qquad T_1 = B_1 = 0 \qquad (0 \leqq z \leqq l) \tag{48}$$

$$M_2 = 0 \qquad T_2' - B_2'' = 0 \qquad (0 \leqq z \leqq l)$$
$$B_{2(0)} = 0, \tag{49}$$

which are orthogonal in the sense of eqn (20), although not normalized. The application of this system of functions to specific problems is treated below in the context of each problem.

Before going to the general case we first cover the common, and technically significant, special case in which the effect of warping is negligible, that is $E\Gamma \ll GKl^2$. This includes narrow rectangular beams, T-sections, etc., as well as I-beams which are fairly narrow in comparison with their length.

In this case $B = 0$, and the boundary condition $\beta'_{(l)} = 0$ is eliminated. The degree of redundancy is $n = 2$, and eqn (49) implies that T_2 is constant. The orthogonality conditions eqns (18) then become

$$\int_0^l \frac{Mz}{EI}\,dz = 0 \tag{50}$$

$$\int_0^l \frac{T}{GK}\,dz = 0. \tag{51}$$

Of these, the first is not available if the right support is simply supported; in that case, however, $Q = 0$.

The expression for Ω^* is obtained by deleting B from eqn (47). Variation with respect to m, T and Q leads, respectively, to the following equations:

$$\lambda\frac{M}{EI} - \frac{T'}{m} = 0 \qquad (0 \leqq z \leqq l) \tag{52}$$

$$\lambda\frac{T}{GK} + \left(\frac{M - Qz}{m}\right)' = 0 \qquad (0 \leqq z \leqq l) \qquad \text{(a)}$$

$$M_{(0)} = 0 \qquad \text{(b)} \tag{53}$$

$$M_{(l)} - Ql = 0 \qquad \text{(c)}$$

$$\int_0^l \frac{T'z}{m}\,dz = 0. \tag{54}$$

It is interesting to note, as before, that the orthogonality conditions eqns (50) and (51) are equivalent, respectively, to eqn (54) and a combination of eqns (53b) and (53c) provided $\lambda \neq 0$. We therefore use eqns (50), (51) and (say) eqn (53b), which provide the necessary equations to determine the two constants of integration and Q, and which guard, at the same time, against the singular case of $\lambda = 0$.

This system of equations can be used to obtain an exact solution to the problem. One of the principal advantages of the complementary energy method, however, is its adaptability to approximate techniques, and for that purpose, and in particular for the purpose of ensuring upper bound solutions, it is necessary that the Timoshenko fraction λ_T have a positive definite numerator. This is accomplished by eliminating one of the two variables and by expressing Ω^* in terms of the remainder.

For example, we may eliminate M by means of eqns (52) while remembering that eqns (50) and (54) are equivalent. Then

$$\Omega^*_{(T)} = \frac{1}{2}\int_0^l \left[\frac{T^2}{GK} - \frac{EI}{\lambda^2}\left(\frac{T'}{m}\right)^2\right] dz \tag{55}$$

subject to the restriction of eqns (51) and (54) (the latter being inoperative for simple support). An exact solution is obtained by incorporating these restrictions through the employment of Lagrangian multipliers. With the unrestricted expression for Ω^* now being given by

$$\Omega^*_{(T)} = \frac{1}{2}\int_0^l \left[\frac{T^2}{GK} - \frac{EI}{\lambda^2}\left(\frac{T'}{m}\right)^2 - 2\mu\frac{T}{GK} + 2\nu\frac{T'z}{m}\right] dz, \tag{56}$$

variation with respect to T yields

$$\begin{aligned} \frac{T}{GK} + \left(\frac{EI}{\lambda^2}\frac{T'}{m^2}\right)' - \frac{\mu}{GK} - \nu\left(\frac{z}{m}\right)' &= 0 \qquad (0 \leqq z \leqq l) \\ \frac{EI}{\lambda^2}\frac{T'}{m^2} + \frac{\nu z}{m} &= 0 \qquad (z = 0, l) \end{aligned} \tag{57}$$

which has to be solved together with eqns (51) and (54). Similarly, again subject to eqns (51) and (54), an approximate Timoshenko fraction is given by

$$\lambda_{cr}^2 = \min_T \lambda_T^2 = \min_T \frac{\int_0^l EI\left(\frac{T'}{m}\right)^2 dz}{\int_0^l \frac{T^2}{GK} dz} \tag{58}$$

which, as pointed out previously, yields an upper bound which is lower, and hence better, than the Rayleigh fraction.

As an example consider the case of constant stiffnesses EI and GK, and consider the (somewhat artificial but simple) case of a beam subjected to two equal major bending moments at its ends, that is, $m = 1$. A straightforward calculation then leads to the well-established result

$$\lambda_{cr} = \frac{4{\cdot}49}{l}\sqrt{EIGK} \equiv 4{\cdot}49D.$$

For an approximate solution we try

$$T = a\left(\frac{z}{l}\right)^2 + \frac{b}{3}\left(\frac{z}{l}\right) + \frac{c}{3}$$

and, in order to satisfy eqns (54) and (51), respectively, we set $b = -4a$ and $c = a$. Then $\lambda_T = 5{\cdot}48D$, which, as expected, is an upper bound to λ_{cr}. On the other hand, with $b = 0$ and $c = -a$ we obtain $\lambda_T = 3{\cdot}87D$, this is a lower bound to λ_{cr}, but an upper bound to the solution (πD) of the corresponding simply supported problem, since in this case eqn (51), but not eqn (54) is satisfied. If eqns (51) is violated, as, e.g. in the case of $c = 0$, $b = -4a$, the approximate solution ($1{\cdot}55D$) is an unacceptable lower bound.

As an alternative we may remove T by means of eqns (53). Since eqn (51) is thereby automatically taken care of (for $\lambda \neq 0$), the complementary energy now becomes

$$\Omega^*_{(M)} = \frac{1}{2}\int_0^l \left[\frac{M^2}{EI} - \frac{GK}{\lambda^2}\left(\frac{M - M_{(l)}(z/l)}{m}\right)'^2\right] dz \tag{59}$$

subject to eqn (50) and to $M_{(0)} = 0$. An appropriate Timoshenko fraction then is

$$\lambda_{cr}^2 = \min_M \lambda_T^2 \equiv \min_M \frac{\int_0^l GK\left(\frac{M - M_{(l)}(z/l)}{m}\right)'^2 dz}{\int_0^l \frac{M^2}{EI} dz - \frac{\left(\int_0^l \frac{Mz}{EI} dz\right)^2}{\int_0^l \frac{z^2}{EI} dz}} \tag{60}$$

in which the last term in the denominator obviates eqns (50). For example, again with constant stiffnesses and $m = 1$, let $M = a(z/l)[1-(z/l)]$. Then $\lambda_T = 5{\cdot}16D$ (an upper bound), whereas, without the modifying term in the denominator, $\lambda_T = 3{\cdot}16D$ (a lower bound, but a close upper bound to the corresponding simply supported problem).

Finally, in order to include the warping effect we return to eqn (47) and, through variation with respect to M, T, B and Q, establish the following system of equations:

$$\lambda\frac{M}{EI} - \frac{1}{m}(T' - B'') = 0 \qquad (0 \leqq z \leqq l) \tag{61}$$

$$\lambda\frac{T}{GK} + \left(\frac{M - Qz}{m}\right)' = 0 \qquad (0 \leqq z \leqq l) \qquad \text{(a)}$$

$$M_{(0)} = 0 \qquad \text{(b)} \qquad (62)$$

$$M_{(l)} - Ql = 0 \qquad \text{(c)}$$

$$\lambda\frac{B}{E\Gamma} + \left(\frac{M - Qz}{m}\right)'' = 0 \qquad (0 \leqq z \leqq l) \qquad \text{(a)}$$

$$\left(\frac{M - Qz}{m}\right)'_{(l)} = \frac{1}{m_{(l)}}[M'_{(l)} - Q] = 0 \qquad \text{(b)} \qquad (63)$$

$$\int_0^l \frac{1}{m}(T' - B'')z \, dz = 0 \tag{64}$$

in which eqns (62), which are identical with eqns (53), have been repeated for the sake of convenience, and in which eqns (62b, c) have been used to simplify eqn (63b). The solution of these equations must be orthogonal to M_1, T_1 and B_1, as defined in eqn (48), that is, it must satisfy eqn (50), and it must also be orthogonal to M_2, T_2 and B_2 as defined in eqn (49).

In order to identify this orthogonality condition we note that, by eqn (49),

$$T_2 - B_2' - Q_2 = 0 \tag{65}$$

in which Q_2 is a constant of integration. If we now introduce the Lagrangian multipliers $\beta'(z)$ and b_0, then

$$\int_0^l \left[\frac{TT_2}{GK} + \frac{BB_2}{E\Gamma} - \beta'(T_2 - B_2' - Q_2)\right] dz - b_0 B_{2(0)} = 0 \tag{66}$$

for all functions $T_2(z)$ and $B_2(z)$. As a consequence†

$$\frac{T}{GK} - \beta' = 0 \qquad (0 \leqq z \leqq l)$$

$$\frac{B}{E\Gamma} - \beta'' = 0 \qquad (0 \leqq z \leqq l)$$

$$\beta_{(l)} - \beta_{(0)} = 0 \qquad (67)$$

$$\beta'_{(l)} = 0.$$

With the elimination of β from eqns (67) the orthogonality conditions become

$$\frac{B}{E\Gamma} - \left(\frac{T}{GK}\right)' = 0 \qquad (0 \leqq z \leqq l)$$

$$\int_0^l \frac{T}{GK} dz = 0 \qquad T_{(l)} = \frac{GK}{l}\int_0^l \frac{zB}{E\Gamma} dz = 0, \qquad (68)$$

which, as usual, admit a kinematic interpretation for $\lambda \neq 0$.

An expression for the Timoshenko fraction is finally obtained by removing T, B and Q

†If β represents the rotation then an interpretation of eqns (67) is obvious. It should be borne in mind, however, that the principal object of the current study is to demonstrate the use of the complementary energy, and the method outlined here is intended to serve as a guide to the solution of other (less obvious) problems.

through eqns (62) and (63). If eqn (50) is once again taken care of through a suitable adjustment of the denominator, then

$$\lambda_{cr}^2 = \min_M \lambda_T^2 \equiv \min_M \frac{\int_0^l \left[GK\left(\frac{M - M_{(l)}(z/l)}{m}\right)'^2 + E\Gamma\left(\frac{M - M_{(l)}(z/l)}{m}\right)''^2\right] dz}{\int_0^l \frac{M^2}{EI} dz - \frac{\left(\int_0^l \frac{Mz}{EI} dz\right)^2}{\int_0^l \frac{z^2}{EI} dz}} \tag{69}$$

subject, once again, to $M_{(0)} = 0$ and $M_{(l)} = lM'_{(l)}$.

To demonstrate the effectiveness of this approach, both exact and approximate, let us again assume the case of constant stiffnesses EI and GK as well as $m = 1$, and introduce $\gamma^2 = E\Gamma/l^2GK = \text{constant}$. Then the solution of eqns (61)–(64) leads to the familiar result

$$\lambda_1 = 4{\cdot}49D[1 + (4{\cdot}49)^2\gamma^2]^{1/2}$$

where, once again, $D = (1/l)\sqrt{EIGK}$. For an approximate solution we try, as we did in the case of eqn (60), $M = a(z/l)(1 - z/l)$; then eqn (69) yields the result

$$\lambda_T = 5{\cdot}16D[1 + 12\gamma^2]^{1/2}$$

which represents an upper bound for small values of γ, but a lower bound for sufficiently large values of γ. The reason for this discrepancy lies in the fact that the assumed function for M violates the second boundary condition. However, with $M = az/l(1 - z/l)^2$ (satisfying both boundary conditions),

$$\lambda_T = 4{\cdot}65D[1 + 30\gamma^2]^{1/2},$$

which constitutes a good upper bound for all values of γ.

5. CONCLUDING REMARKS

The systematic development, and efficacy of application, of the complementary energy method has been demonstrated in connection with the problem of elastic instability. Other eigenvalue problems are, of course, equally amenable to this type of approach. For example, the beam vibration problem, with a simple support at the left end and fixed support at the right end, can be expressed in the form

$$\omega^2\Omega^{**} = \frac{1}{2}\int_0^l \left(\omega^2\frac{M^2}{EI} - 2Mv'' + \rho v^2\right) dz + q_0 v_{(0)} - q_l v_{(l)} + m_l v'_{(l)} \tag{70}$$

in which ω^2, the square of the frequency, represents the eigenvalue and ρ the linear mass density. Variation of eqn (70) with respect to v leads to the equations of equilibrium

$$\begin{aligned} &M'' - \rho v = 0 \qquad (0 \leqq z \leqq l) \\ &M_{(0)} = 0 \end{aligned} \tag{71}$$

and the complementary energy then becomes

$$\begin{aligned} &\Omega^* = \frac{1}{2}\int_0^l \left(\frac{M^2}{EI} - \frac{M''^2}{\omega^2\rho}\right) dz \\ &M_{(0)} = 0 \\ &\int_0^l \frac{M}{EI} z \, dz = 0. \end{aligned} \tag{72}$$

For example, for constant EI and ρ, and with $\omega^2 = \phi^4 EI/\rho l^4$, variation of Ω^* with respect to M results in the compatibility conditions

$$M^{iv} - \frac{\phi^4}{l^4} M = 0 \qquad (0 \leqq z \leqq l)$$

$$M_{(0)} = M''_{(0)} = M''_{(l)} = M'''_{(l)} = 0 \tag{73}$$

whose characteristic equation is tan ϕ = tanh ϕ. The approximate solution is given by

$$\omega_T^2 = \frac{\int_0^l \frac{M''^2}{\rho} \mathrm{d}z}{\int_0^l \frac{M^2}{EI} \mathrm{d}z - \left[\left(\int_0^l \frac{Mz}{EI} \mathrm{d}z\right)^2 \Big/ \int_0^l \frac{z^2}{EI} \mathrm{d}z\right]}. \tag{74}$$

For example, if $M = a \sin \pi z/l$, then $\omega_T^2 = 237\ EI/\rho l^4$, which represents a fairly close upper bound to the exact solution.

The general availability, as well as the numerical efficacy, of the complementary energy approach in solving structural eigenvalue problems has therefore been demonstrated. Its primary advantage appears to reside in the fact that the order of the equations may be reduced and, concurrently, the accuracy of approximate solutions enhanced. Along a similar vein it appears reasonable that the method should also lend itself readily to a finite element approximation, including the use of simpler elements associated with simpler boundary and continuity conditions and hence shape functions.

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